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A SEMIGROUPS THEORY APPROACH TO A MODEL OF SUSPENSION BRIDGES

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Abstract. In this paper we study the existence and uniqueness of the weak solution of a mathematical model that describes the nonlinear oscillations of a suspension bridge. This model is given by a system of partial differential equations with damping terms. The main tool used to show this is the C_0-semigroup theory extending the results of Aassila [1].

1. Introduction

Since the collapse of the Tacoma Narrows Bridge on November 7, 1940, several mathematical models have been proposed to study the oscillations of the bridge. Lazer and McKenna proposed a model governed by a coupled system of PDEs which takes into account the coupling provided by the stays (ties) connecting the suspension (main) cable to the deck of the road bed. In this model the coupling is nonlinear (for more details see [12]).

In [3] Ahmed and Harbi used the model proposed by Lazer and McKenna to do a detailed study of various types of damping. Also, they presented an abstract approach which allows the study of the regularity of solutions of these models.

The model of suspension bridges is given by the system of partial differential equations

$$\begin{cases} m_b z_{tt} + \alpha z_{xxxx} - F(y-z) = f_1(z_t),\ x \in \Omega,\ t \geqslant 0, \\ m_c y_{tt} - \beta y_{xx} + F(y-z) = f_2(y_t),\ x \in \Omega,\ t \geqslant 0, \\ z(0,t) = z(l,t) = 0, z_x(0,t) = z_x(l,t) = 0, \\ y(0,t) = y(l,t) = 0, \\ z(x,0) = z_1(x),\ z_t(x,0) = z_2(x),\ x \in \Omega, \\ y(x,0) = y_1(x),\ y_t(x,0) = y_2(x),\ x \in \Omega. \end{cases} \tag{1.1}$$

Here we denote by Ω the interval $(0,l)$. See [3] and [12] for the physical interpretations of the parameters α, β, the variables y, z and the boundary conditions respectively. As described in [3] the function F represents the restraining force experienced by both the road bed and the suspension cable as transmitted through the tie lines (stays), thereby producing the coupling between these two.

The functions f_1 and f_2 represent external forces as well as non-conservative forces, which generally depend on time, the constants m_b, m_c, α, β are positive and $F : \mathbb{R} \to \mathbb{R}$ is a function with $F(0) = 0$ (F can be linear or not), see [3]. The interested reader is also refereed to the works of Drábek et. al [6, 7] and Holubová [11] where other models for the oscillations of the bridge are studied.

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The aim of this work is to study the existence and uniqueness of weak solutions for (1.1). To do this we make use of the *semigroup theory.* This allows us to do it in a much simpler way without using maximal monotone operators theory as in [10, Theorem 1] or the Galerkin approach as in [3, Theorem 4.4].

In Section 2, we study the existence and uniqueness of weak solutions of the linear model of suspension bridges, i.e., when $F(\xi) = k\xi$ and $f_1 = f_2 = 0$ (see, for instance, [3]). The case $F(\xi) = k\xi$ and $f_1 \neq f_2 \neq 0$ was considered by Aassila in [1]. We consider the nonlinear model in Section 3.

2. Linear abstract model

The linear model is obtained through the bed support bridge tied with cords connected to two main cables placed symmetrically (suspended), one above and one below the bed of the bridge. In the absence of external forces ($f_1 = f_2 = 0$), the linear dynamic of suspension bridge around the equilibrium position can be described by the following system of linear coupled EDP's

$$\begin{cases} m_b z_{tt} + \alpha z_{xxxx} - k(y-z) = 0, \quad x \in \Omega,\ t \geqslant 0, \\ m_c y_{tt} - \beta y_{xx} + k(y-z) = 0, \quad x \in \Omega,\ t \geqslant 0, \\ z(0,t) = z(l,t) = 0, z_x(0,t) = z_x(l,t) = 0, \\ y(0,t) = y(l,t) = 0, \\ z(x,0) = z_1(x),\ z_t(x,0) = z_2(x),\ x \in \Omega, \\ y(x,0) = y_1(x),\ y_t(x,0) = y_2(x),\ x \in \Omega. \end{cases} \tag{2.1}$$

Here, $F(\xi) = k\xi$, where k denotes the stiffness coefficient of the cables connecting the bridge to the bed and suspended cable.

2.1. **Existence and uniqueness of solution.** Let us denote for $H = L^2(\Omega) \times L^2(\Omega)$, $V = H_0^2(\Omega) \times H_0^1(\Omega)$, and $W = (H^4(\Omega) \cap H_0^2(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))$ the Hilbert spaces endowed with scalar products

$$\begin{aligned} \langle(\phi_1,\psi_1),(\phi_2,\psi_2)\rangle_H &:= \int_\Omega (m_b\phi_1\phi_2 + m_c\psi_1\psi_2)dx, \\ \langle(\phi_1,\psi_1),(\phi_2,\psi_2)\rangle_V &:= \int_\Omega (\alpha\Delta\phi_1\Delta\phi_2 + \beta\nabla\psi_1\nabla\psi_2 + k(\psi_1-\phi_1)(\psi_2-\phi_2))dx, \\ \langle(\phi_1,\psi_1),(\phi_2,\psi_2)\rangle_W &:= \int_\Omega (\zeta\Delta^2\phi_1\Delta^2\phi_2 + \theta\Delta\nabla\phi_1\Delta\nabla\phi_2 + \xi\Delta\psi_1\Delta\psi_2)dx \\ &\quad + \langle(\phi_1,\psi_1),(\phi_2,\psi_2)\rangle_V, \quad \zeta,\theta,\xi > 0; \end{aligned}$$

and with their respective norms

$$\|(\phi,\psi)\|_H^2 := \int_\Omega (m_b|\phi|^2 + m_c|\psi|^2)dx,$$
$$\|(\phi,\psi)\|_V^2 := \int_\Omega (\alpha|\Delta\phi|^2 + \beta|\nabla\psi|^2 + k|\psi-\phi|^2)dx,$$
$$\|(\phi,\psi)\|_W^2 := \int_\Omega (\zeta|\Delta^2\phi|^2 + \theta|\Delta\nabla\phi|^2 + \xi|\Delta\psi|^2)dx + \|(\phi,\psi)\|_V^2.$$

It is well know that norm $\|(\cdot,\cdot)\|_V^2$ defined in V is equivalent to the usual norm of $H^2(\Omega)\times H^1(\Omega)$ and, consequently the norm $\|(\cdot,\cdot)\|_W^2$ defined in W is equivalent to the norm of $H^4(\Omega)\times H^2(\Omega)$. Therefore, by the Sobolev embeddings in [5, p. 23], we have the embeddings dense and compact $W\subset V\subset H$. Identifying H with its dual H', we obtain $W\subset V\subset H = H'\subset V'\subset W'$ with embeddings dense and compact.

Let the bilinear form $\mathfrak{a} : V\times V\to\mathbb{R}$ be given by

$$\mathfrak{a}(u,\tilde{u}) = \alpha\langle\Delta u_1,\Delta\tilde{u}_1\rangle_{L^2(\Omega)} + \beta\langle\nabla u_2,\nabla\tilde{u}_2\rangle_{L^2(\Omega)} + k\langle u_2-u_1,\tilde{u}_2-\tilde{u}_1\rangle_{L^2(\Omega)}, \tag{2.2}$$

where $u=(u_1,u_2)$, $\tilde{u}=(\tilde{u}_1,\tilde{u}_2)\in V$. To simplify the notation, we use $\langle\cdot,\cdot\rangle_{L^2(\Omega)} = \langle\cdot,\cdot\rangle$ and $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$.

Lemma 2.1. *The bilinear form $\mathfrak{a}$ is continuous, symmetric and coercive.*

Proof. For $u=(u_1,u_2), v=(v_1,v_2)\in V$, we have

$$\begin{aligned}
|\mathfrak{a}(u,v)|^2 &\leqslant \alpha^2\|\Delta u_1\|^2\|\Delta v_1\|^2 + \beta^2\|\nabla u_2\|^2\|\nabla v_2\|^2 + k^2\|u_2-u_1\|^2\|v_2-v_1\|^2\\
&\quad + 2\alpha\beta\|\Delta u_1\|\|\Delta v_1\|\|\nabla u_2\|\|\nabla v_2\| + 2k\alpha\|\Delta u_1\|\|\Delta v_1\|\|u_2-u_1\|\|v_2-v_1\|\\
&\quad + 2k\beta\|\nabla u_2\|\|\nabla v_2\|\|u_2-u_1\|\|v_2-v_1\|\\
&\leqslant \alpha^2\|\Delta u_1\|^2\|\Delta v_1\|^2 + \beta^2\|\nabla u_2\|^2\|\nabla v_2\|^2 + k^2\|u_2-u_1\|^2\|v_2-v_1\|^2\\
&\quad + \alpha\beta\|\Delta u_1\|^2\|\nabla v_2\|^2 + \alpha\beta\|\Delta v_1\|^2\|\nabla u_2\|^2 + k\alpha\|\Delta u_1\|^2\|v_2-v_1\|^2 +\\
&\quad + k\alpha\|\Delta v_1\|^2\|u_2-u_1\|^2 + k\beta\|\nabla u_2\|^2\|v_2-v_1\|^2 + k\beta\|\nabla v_2\|^2\|u_2-u_1\|^2\\
&= \|(u_1,u_2)\|_V^2\|(v_1,v_2)\|_V^2.
\end{aligned}$$

Thus, $\mathfrak{a}(u,v)\leqslant\|u\|_V\|v\|_V$, and this shows that $\mathfrak{a}$ is continuous. The symmetry of $\mathfrak{a}$ is immediate. For the last, $\mathfrak{a}(u,u) = \alpha\|\Delta u_1\|^2 + \beta\|\nabla u_2\|^2 + k\|u_2-u_1\|^2 = \|u\|_V^2$, for all $u=(u_1,u_2)\in V$, and thus we have the coercivity of $\mathfrak{a}$. ☐

From Lemma 2.1, there exists a linear operator $C\in\mathscr{L}(V,V')$ such that $\mathfrak{a}(u,v) = \langle Cu,v\rangle_{V',V}$, $\forall u,v\in V$.

For $u=(z,y)$, the system (2.1) can be written as

$$\begin{cases} u_{tt} + Cu = 0,\ (x,t)\in\Omega\times(0,\infty)\\ u = u_x = 0,\ \text{on } \partial\Omega\times(0,\infty)\\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x),\ x\in\Omega, \end{cases} \tag{2.3}$$

where $Cu = (a\Delta^2 z - p(y-z), -b\Delta y + q(y-z))$, $a^2 = \frac{\alpha}{m_b}$, $b^2 = \frac{\beta}{m_c}$, $p = \frac{k}{m_b}$, $q = \frac{k}{m_c}$, $u_0(x) = (z_1(x), y_1(x))$, and $v_0(x) = (z_2(x), y_2(x))$.

With this notation we can see the problem (2.3) as second order ODE in H,

$$\begin{cases} u_{tt} + Cu = 0, \ t \in [0,\infty), \\ u(0) = u_0, \ u_t(0) = v_0, \end{cases} \tag{2.4}$$

where the operator $C : D(C) \subset H \to H$ has domain $D(C)$ given by

$$D(C) = \{u = (z,y) \in H : \Delta^2 z, \Delta y \in L^2(\Omega), \ z = \nabla z = y = 0 \text{ on } \partial\Omega\}. \tag{2.5}$$

Consequently, we have for the operator C,

$$D(C) = [H^4(\Omega) \cap H_0^2(\Omega)] \times [H^2(\Omega) \cap H_0^1(\Omega)] = W. \tag{2.6}$$

Proposition 2.2. *The operator $-C$ is infinitesimal generator of a C_0-semigroup contractions in H.*

Proof. Let $u = (z,y) \in D(C)$, then $\langle -Cu, u\rangle_{V',V} = -\langle Cu, u\rangle_{V',V} = -\mathfrak{a}(u,u) = -\|u\|_V^2 \leqslant 0$. This show that $-C$ is dissipative.

Now, for $u = (z,y), \tilde{u} = (\tilde{z}, \tilde{y}) \in D(C)$ we have

$$\langle -Cu, \tilde{u}\rangle_{V',V} = -\langle Cu, \tilde{u}\rangle_{V',V} = -\mathfrak{a}(u,\tilde{u}) = -\mathfrak{a}(\tilde{u},u) = -\langle C\tilde{u}, u\rangle_{V',V} = \langle u, -C\tilde{u}\rangle_{V',V},$$

thus, $-C$ is symmetric.

Let $u_n = (z_n, y_n) \in D(C)$ be such that $u_n \to u = (z,y)$ and $Cu_n \to (\eta, \zeta)$. Then, $z_n \to z$ in $H^4(\Omega) \cap H_0^2(\Omega)$, $y_n \to y$ in $H^2(\Omega) \cap H_0^1(\Omega)$, $a^2\Delta^2 z_n - p(y_n - z_n) \to \eta$ and $-b^2\Delta y_n + q(y_n - z_n) \to \zeta$ in $L^2(\Omega)$. We know the operators Δ and Δ^2 with domain $H^2(\Omega) \cap H_0^1(\Omega)$ and $H^4(\Omega) \cap H_0^2(\Omega)$ respectively, are closed (see [9, Lema 18.1]). Thus, by uniqueness of limits we have $a^2\Delta^2 z - p(y-z) = \eta$ and $-b^2\Delta y - q(y-z) = \zeta$, that is, $Cu = (\eta,\zeta)$ and $u \in D(C)$. Therefore C is closed. Now, by (2.6) and Corollary 4.4 in [13, p. 15] follows that $-C$ infinitesimal generator of a C_0-semigroup in H. □

Notice that in the equation (2.4) we are looking for u as a function of t taking values on H, i.e., $[0,\infty) \ni t \mapsto u(t) \in H$ with $u(t)(x) = u(x,t)$, $x \in \Omega$.

The problem (2.4) can be written as a first order EDO abstract

$$\begin{cases} u_t - v = 0 \\ v_t + Cu = 0 \end{cases} \tag{2.7}$$

with the boundary condition $v = u_t = (z_t, y_t) = 0$ on $\partial\Omega \times (0,\infty)$.

Let us denote for $\mathcal{H} = V \times H$ the Hilbert space endowed with the inner product $\big\langle(\phi_1,\psi_1),(\phi_2,\psi_2)\big\rangle_{\mathcal{H}} = \langle\phi_1,\phi_2\rangle_V + \langle\psi_1,\psi_2\rangle_H$.

For $U = (u,v)$ the system (2.7) can be written as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \dot{U} + \mathbf{A}U = 0, \quad t \in (0,\infty) \\ U(0) = U_0, \end{cases} \tag{2.8}$$

where $U_0 = (u_0, v_0)$, $\mathbf{A} : D(\mathbf{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by $\mathbf{A}U = (-v, Cu)$ and

$$\begin{aligned} D(\mathbf{A}) &= \{U = (u, v) \in V \times H : (-v, Cu) \in V \times H\} \\ &= \{U = (u, v) \in V \times V : Cu \in H\} . \end{aligned}$$

Lemma 2.3. *For the operator* $\mathbf{A}$ *holds that* $D(\mathbf{A}) = W \times V$ *and* $D(A)$ *is dense in* $\mathcal{H}$.

Proof. See the details in [1, Lemma 2.3]. □

Lemma 2.4. *The operator* $-\mathbf{A}$ *is the infinitesimal generator of a* C_0*-semigroup of contractions in* $\mathcal{H}$.

Proof. Let $U, \tilde{U} \in D(\mathbf{A})$ then

$$\begin{aligned} \langle -\mathbf{A}U, \tilde{U}\rangle_{\mathcal{H}} &= \big\langle (v, -Cu), (\tilde{u}, \tilde{v})\big\rangle_{\mathcal{H}} = \langle v, \tilde{u}\rangle_V + \langle -Cu, \tilde{v}\rangle_H \\ &= \big\langle (z_t, y_t), (\tilde{z}, \tilde{y})\big\rangle_V + \big\langle (-a^2\Delta^2 z + p(y-z), b^2\Delta y - q(y-z)), (\tilde{z}_t, \tilde{y}_t)\big\rangle_H \\ &= \int_\Omega \big[\alpha\Delta z_t \Delta\tilde{z} + \beta\nabla y_t \nabla\tilde{y} + k(y_t - z_t)(\tilde{y} - \tilde{z}) - \alpha\Delta^2 z\tilde{z}_t + k(y-z)\tilde{z}_t \\ &\quad + \beta\Delta y\tilde{y}_t - k(y-z)\tilde{y}_t\big]\,dx \\ &= \int_\Omega \big[-\alpha\Delta z\Delta\tilde{z}_t - \beta\nabla y\nabla\tilde{y}_t - k(\tilde{y}_t - \tilde{z}_t)(y-z)\big]\,dx \\ &\quad + \int_\Omega \big[\alpha z_t\Delta^2\tilde{z} - \beta y_t\Delta\tilde{y} + k(y_t - z_t)(\tilde{y} - \tilde{z})\big]\,dx \\ &= \big\langle (z, y), -(\tilde{z}_t, \tilde{y}_t)\big\rangle_V + \big\langle (z_t, y_t), (a^2\Delta^2\tilde{z} - p(\tilde{y} - \tilde{z}), -b^2\Delta\tilde{y} + q(\tilde{y} - \tilde{z}))\big\rangle_H \\ &= \langle u, -\tilde{v}\rangle_V + \langle v, C\tilde{u}\rangle_H = \langle U, \mathbf{A}\tilde{U}\rangle_{\mathcal{H}}. \end{aligned}$$

Thus, $(-\mathbf{A})^* = \mathbf{A}$. Analogously to what we did before, we get $\langle -\mathbf{A}U, U\rangle_{\mathcal{H}} = 0$. Therefore, $-\mathbf{A}$ and $(-\mathbf{A})^*$ are dissipative.

Now, let $U_n = (u_n, v_n) \in D(\mathbf{A})$ be such that $U_n \to U = (u, v)$ and $\mathbf{A}U_n = (-v_n, Cu_n) \to (\tilde{u}, \tilde{v})$. Then, $u_n \to u$ in V, $v_n \to v$ in H, $v_n \to -\tilde{u}$ in V and $Cu_n \to \tilde{v}$ in H. From this, we have $\tilde{u} = -v \in V$. Since C is closed, it follows that $Cu = \tilde{v}$ and $u \in D(C) = W$. Thus, $(\tilde{u}, \tilde{v}) = (-v, Cu) = \mathbf{A}U$ and $U \in W \times V = D(\mathbf{A})$. Therefore $\mathbf{A}$ is closed.

Now, by Lemma 2.3 and Corollary 4.4 [13, p. 15] it follows that $-\mathbf{A}$ is infinitesimal generator of a C_0-semigroup of contractions in $\mathcal{H}$. □

Theorem 2.5 (Existence and uniqueness)**.** *Given* $(z_1, y_1, z_2, y_2) \in V \times H$*, the problem* (2.1) *has a unique weak solution*

$$(z, y) \in C([0, \infty), V) \cap C^1([0, \infty), H).$$

Moreover, if $(z_1, y_1, z_2, y_2) \in W \times V$*, the*

$$(z, y) \in C([0, \infty), W) \cap C^1([0, \infty), V) \cap C^2([0, \infty), H).$$

Proof. The problem (2.1) is equivalent to the problem (2.8) with $U_0 = (z_1, y_1, z_2, y_2) \in \mathcal{H}$. We know from Lemma 2.4 that $-\mathbf{A}$ is infinitesimal generator of a C_0-semigroup contractions in $\mathcal{H}$ and by the Sobolev embeddings we have $\operatorname{int}(D(\mathbf{A})) \neq \emptyset$. Thus, by Theorem 3.3 in [4, p.62], there is a unique solution $U \in C([0,\infty),\mathcal{H})$. Therefore,

$$\begin{aligned}(u,v) \in C([0,\infty), V \times H) &\Rightarrow u \in C([0,\infty), V),\ u_t \in C([0,\infty), H)\\ &\Rightarrow (z,y) \in C([0,\infty), V) \cap C^1([0,\infty), H).\end{aligned}$$

This prove the first part of the theorem.

On the other hand, if $U_0 \in D(\mathbf{A}) = W \times V$ and $-\mathbf{A}$ is infinitesimal generator of a C_0-semigroup contractions in $\mathcal{H}$ then we have a unique solution (Proposition 6.2 in [8, p. 110])

$$U \in C([0,\infty), D(\mathbf{A})) \cap C^1([0,\infty),\mathcal{H}).$$

Thus,

$$\begin{aligned}&(u,v) \in C([0,\infty), W \times V) \cap C^1([0,\infty), V \times H)\\ &\Rightarrow u \in C([0,\infty), W),\ u_t \in C([0,\infty), V) \text{ and } u \in C^1([0,\infty), V),\ u_t \in C^1([0,\infty), H)\\ &\Rightarrow u \in C([0,\infty), W) \cap C^1([0,\infty), V) \cap C^2([0,\infty), H)\\ &\Rightarrow (z,y) \in C([0,\infty), W) \cap C^1([0,\infty), V) \cap C^2([0,\infty), H).\end{aligned}$$

This proves the second part of the theorem. □

3. Nonlinear abstract model

In this section we consider the general problem (1.1) which can be seen as an abstract ODE in a suitable Hilbert space. The abstract setting has many advantages as we can see below. We first write the equation of the problem (1.1) as follows

$$\begin{cases} z_{tt} + a^2\Delta^2 z = F_1(t,x,y,z), & x \in \Omega, \quad t > 0,\\ y_{tt} - b^2\Delta y = F_2(t,x,y,z), & x \in \Omega, \quad t > 0. \end{cases} \tag{3.1}$$

Here

$$\begin{aligned} F_1(t,x,y,z) &= \frac{1}{m_b}(\,F(y-z) + f_1(z_t)),\\ F_2(t,x,y,z) &= \frac{1}{m_c}(-F(y-z) + f_2(y_t)). \end{aligned} \tag{3.2}$$

Let H be the Hilbert space as before and consider V given by $V = H^2(\Omega) \times H_0^1(\Omega)$ endowed with the inner product and norm given by

$$\langle(\phi_1,\phi_2),(\psi_1,\psi_2)\rangle_V := \big\langle(\phi_1)_{xx},(\psi_1)_{xx}\big\rangle_{L^2(\Omega)} + \big\langle(\phi_2)_x,(\psi_2)_x\big\rangle_{L^2(\Omega)}. \tag{3.3}$$

By Poincaré's inequality the norms $\|v\|^2_{H^m(\Omega)} = \sum_{|\alpha|\leqslant m}\|D^\alpha v\|_{L^2(\Omega)}$ and $\|v\|^2_{H_0^m(\Omega)} = \sum_{|\alpha|=m}\|D^\alpha v\|_{L^2(\Omega)}$ are equivalents and thus V is a Hilbert space. Note that the embedding $V \hookrightarrow H$ is continuous, dense and compact. If V' denotes the dual topological of V and identifying H with its dual we have the inclusions $V \hookrightarrow H \hookrightarrow V'$ compact. Note that $V' = H^{-2}(\Omega) \times H^{-1}(\Omega)$, where $H^{-s}(\Omega)$, $s > 0$, denotes the Sobolev's space with negative exponent, for more details see [2].

Consider the bilinear form $\mathfrak{c} : V \times V \to \mathbb{R}$ defined by

$$\mathfrak{c}(u, v) = a^2 \langle \Delta u_1, \Delta v_1 \rangle_{L^2(\Omega)} + b^2 \langle \nabla u_2, \nabla v_2 \rangle_{L^2(\Omega)}, \; u = (u_1, u_2), v = (v_1, v_2). \tag{3.4}$$

Lemma 3.1. *The bilinear form $\mathfrak{c}$ is continuous, symmetric and coercive.*

Proof. If $d = a^2 + b^2$ we have

$$\begin{aligned}\mathfrak{c}^2(u, v) &\leqslant a^4 \|\Delta u_1\|^2 \|\Delta v_1\|^2 + b^4 \|\nabla u_2\|^2 \|\nabla v_2\|^2 + 2a^2 b^2 \|\Delta u_1\| |\Delta v_1\| \|\nabla u_2\| \|\nabla v_2\| \\ &\leqslant a^4 \|\Delta u_1\|^2 \|\Delta v_1\|^2 + b^4 \|\nabla u_2\|^2 \|\nabla v_2\|^2 + a^4 \|\Delta u_1\|^2 \|\nabla v_2\|^2 + b^4 \|\Delta v_1\|^2 \|\nabla u_2\|^2 \\ &\leqslant (a^2 + b^2)^2 \|u\|_V^2 \|v\|_V^2.\end{aligned}$$

Then $\mathfrak{c}(u, v) \leqslant d\|u\|_V \|v\|_V$, for all $u, v \in V$, and thus we have that $\mathfrak{c}$ is continuous. The symmetric property is obvious. Finally, denoting $d_0 = \min\{a^2, b^2\}$ we have

$$\begin{aligned}\mathfrak{c}(u, u) &\geqslant \min\{a^2, b^2\}\big(\langle \Delta u_1, \Delta u_1 \rangle_{L^2(\Omega)} + \langle \nabla u_2, \nabla u_2 \rangle_{L^2(\Omega)}\big) \\ &= d_0(\|\Delta u_1\|^2_{L^2(\Omega)} + \|\nabla u_2\|^2_{L^2(\Omega)}\big) = d_0 \|u\|_V^2, \quad \forall\, u \in V;\end{aligned}$$

i.e., $\mathfrak{c}$ is coercive. □

From Lemma 3.1, there exists a linear operator $A \in \mathscr{L}(V, V')$ such that $\mathfrak{c}(u, v) = \langle Au, v \rangle_{V',V}$, for all $u, v \in V$.

The operator $A : D(A) \subset H \to H$ is the realization of the operator

$$Au = (a^2 \Delta^2 u_1, -b^2 \Delta u_2) \tag{3.5}$$

with the boundary condition given in (1.1) and the domain given by

$$D(A) = \{(u_1, u_2) \in H : \Delta^2 u_1, \Delta u_2 \in L^2(\Omega), \; u_1 = \nabla u_1 = u_2 = 0 \text{ in } \partial\Omega\}.$$

Let $t \geqslant 0$ be and consider $u(t) = (u_1(t), u_2(t)) = (z(t, .), y(t, .))$ where the components are functions defined in Ω. Also, consider the operator $\widetilde{F} : \mathbb{R}_0^+ \times H \to H$ given by

$$\widetilde{F}(t, u) = \big(F_1(t, ., u_1(t, .), u_2(t, .)), F_2(t, ., u_1(t, .), u_2(t, .))\big). \tag{3.6}$$

Thus, we can write the system (3.1) as the following abstract second order ODE in the Hilbert space H

$$\begin{cases} u_{tt} + Au = \widetilde{F}(t, u), \quad t \geqslant 0, \\ u(0) = u_0, \; u_t(0) = v_0, \end{cases} \tag{3.7}$$

where $\{u_0, v_0\}$ are given by the initial conditions in (1.1).

It is not difficult to see that

$$D(A) = [H^4(\Omega) \cap H_0^2(\Omega)] \times [H^2(\Omega) \cap H_0^1(\Omega)] = W. \tag{3.8}$$

Proposition 3.2. *The operator $-A$ is the infinitesimal generator of a C_0-semigroup of contractions $\{e^{-At} : t \geqslant 0\}$ in H.*

Proof. Let $u = (z, y) \in D(A)$ be, then $\langle -Au, u\rangle_{V',V} = -\mathfrak{c}(u, u) \leqslant -d_0\|u\|_V^2 \leqslant 0$. That is $-A$ is dissipative.

Since the bilinear form $\mathfrak{c}$ is symmetric, it follows that $(-A)^* = -A$. Now, from Proposition 2.2 it follows that A is closed. Finally, by (3.8) and Corollary 4.4 [13, p. 15] the result follows. $\square$

3.1. **Existence and uniqueness of solution.** Consider the Hilbert space $\mathcal{H} = V \times H$ endowed with the inner product $\langle(\phi_1, \phi_2), (\psi_1, \psi_2)\rangle_{\mathcal{H}} := \mathfrak{c}(\phi_1, \psi_1) + \langle\phi_2, \psi_2\rangle_H$. Thus, we can set the problem (3.7) in $\mathcal{H}$ as

$$\begin{cases} \dot{U}(t) + \mathcal{A}U(t) = \mathcal{F}(t, U(t)), \quad t \geqslant 0, \\ U(0) = U_0, \end{cases} \tag{3.9}$$

where $U = (u, u_t) = (u, v)$, $U_0 = (u_0, v_0)$, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by $\mathcal{A}(u, v) = (-v, Au)$ with

$$\begin{aligned} D(\mathcal{A}) &= \{U = (u, v) \in V \times H : (-v, Au) \in V \times H\} \\ &= \{U = (u, v) \in V \times V : Au \in H\} \end{aligned}$$

and the nonlinear operator $\mathcal{F} : I \times \mathcal{H} \to \mathcal{H}$ given by $\mathcal{F}(t, U) = (0, \widetilde{F}(t, u))$.

Lemma 3.3. *The domain of $\mathcal{A}$ is given by $D(\mathcal{A}) = W \times V$ and $D(\mathcal{A})$ is dense $\mathcal{H}$.*

Proof. Follows from Lemma 2.3. $\square$

Proposition 3.4. *The operator $-\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $\{e^{-\mathcal{A}t} : t \geqslant 0\}$ in the Hilbert space $\mathcal{H}$.*

Proof. Let (u_n, v_n) be a sequence in $D(\mathcal{A})$ such that $(u_n, v_n) \to (u, v)$ and $\mathcal{A}(u_n, v_n) \to (\widetilde{u}, \widetilde{v})$. Then, $u_n \to u$ in W, $v_n \to v$ in V, $-v_n \to \widetilde{u}$ in V and $Au_n \to \widetilde{v}$ in H. From this we have $\widetilde{u} = -v$. Since $u_n \in D(A)$ and A is a closed operator we have $u \in D(A)$ and $Au = \widetilde{v}$. Then, $(\widetilde{u}, \widetilde{v}) = (-v, Au) = \mathcal{A}(u, v)$. Thus $\mathcal{A}$ is closed. By Lemma 3.3, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Now, for $U = (u, v)$, $\widetilde{U} = (\widetilde{u}, \widetilde{v}) \in D(\mathcal{A})$, we have

$$\begin{aligned} \langle -\mathcal{A}U, \widetilde{U}\rangle_{\mathcal{H}} &= \big\langle (v, -Au), (\widetilde{u}, \widetilde{v})\big\rangle_{\mathcal{H}} = \mathfrak{c}(v, \widetilde{u}) + \langle -Au, \widetilde{v}\rangle_H \\ &= \mathfrak{c}((z_t, y_t), (\widetilde{z}, \widetilde{y})) + \big\langle (-a^2\Delta^2 z, b^2\Delta y), (\widetilde{z}_t, \widetilde{y}_t)\big\rangle_H \\ &= a^2\int_\Omega \Delta z_t \Delta\widetilde{z}dx + b^2\int_\Omega \nabla y_t \nabla\widetilde{y}dx - a^2\int_\Omega \Delta^2 z\widetilde{z}_t dx + b^2\int_\Omega \Delta y\widetilde{y}_t dx \\ &= a^2\int_\Omega z_t\Delta^2\widetilde{z}dx - b^2\int_\Omega y_t\Delta\widetilde{y}dx - a^2\int_\Omega \Delta z\Delta\widetilde{z}_t dx - b^2\int_\Omega \nabla y\nabla\widetilde{y}_t dx \\ &= \big\langle v, A\widetilde{u}\big\rangle_H + \mathfrak{c}(u, -\widetilde{v}) = \big\langle (u, v), (-\widetilde{v}, A\widetilde{u})\rangle_{\mathcal{H}} = \langle U, \mathcal{A}\widetilde{U}\rangle_{\mathcal{H}}. \end{aligned}$$

From this we have $(-\mathcal{A})^* = \mathcal{A}$, and analogously we have $\langle -\mathcal{A}U, U\rangle_{\mathcal{H}} = 0$. Thus, $-\mathcal{A}$ and $(-\mathcal{A})^*$ are dissipative. Now, from Corollary 4.4 [13, p. 15] the result follows. $\square$

Theorem 3.5. *Assume that F, f_1 and f_2 satisfy*
(i) *F, f_1 and f_2 are of class C^1 with $F(0)=0$, $f_1(0)=0$ and $f_2(0)=0$.*
(ii) *F, f_1 and f_2 are locally Lipschitz continuous with constants M, c_1 and c_2, respectively.*
(iii) *$|F(s)|^2 \leqslant 1+N|s|^2$, $|f_1(s)|^2 \leqslant 1+c_3|s|^2$ and $|f_2(s)|^2 \leqslant 1+c_4|s|^2$, for all $s \in \mathbb{R}$ and some positive constants N, c_3 and c_4.*

Then, for each $(z_1, y_1, z_2, y_1) \in V \times H$ the problem (1.1) *has a unique weak solution $(z,y) \in C([0,+\infty), V) \cap C^1([0,+\infty), H)$.*

Proof. If $U \in B_r = \{\eta \in \mathcal{H} : \|\eta\|_{\mathcal{H}} \leqslant r\}$ then $\mathfrak{c}(u,u) + \|v\|_H^2 \leqslant r^2$. Since $\mathfrak{c}$ is coercive and $V \hookrightarrow H$ it follows that $\|z\|_{L^2(\Omega)}, \|y\|_{L^2(\Omega)}, \|z_t\|_{L^2(\Omega)}, \|y_t\|_{L^2(\Omega)} \leqslant r$. Similarly, if $\tilde{U} \in B_r$ we obtain the same estimates. Thus, for $U=(u,v), \tilde{U} = (\tilde{u}, \tilde{v}) \in \mathcal{H}$ we have

$$\begin{aligned}
\|\mathcal{F}(t,U) - \mathcal{F}(t,\tilde{U})\|_{\mathcal{H}}^2 &= \mathfrak{c}(0,0) + \|\widetilde{F}(t,\phi) - \widetilde{F}(t,\psi)\|_H^2 \\
&\leqslant 2(m_b^{-2} + m_c^{-2})\|F(y-z) - F(\tilde{y} - \tilde{z})\|_{L^2(\Omega)}^2 \\
&\quad + 2m_b^{-2}\|f_1(z_t) - f_1(\tilde{z}_t)\|_{L^2(\Omega)}^2 + 2m_c^{-2}\|f_2(y_t) - f_2(\tilde{y}_t)\|_{L^2(\Omega)}^2 \\
&\leqslant 4(m_b^{-2} + m_c^{-2})M^2\left[\|y-\tilde{y}\|_{L^2(\Omega)}^2 + \|z - \tilde{z}\|_{L^2(\Omega)}^2\right] \\
&\quad + 2m_b^{-2}c_1^2\|z_t - \tilde{z}_t\|_{L^2(\Omega)}^2 + 2m_c^{-2}c_2^2\|y_t - \tilde{y}_t\|_{L^2(\Omega)}^2 \\
&\leqslant \delta_0\|u - \tilde{u}\|_H^2 + \delta_1\|v - \tilde{v}\|_H^2 \\
&\leqslant \delta_0 d_0^{-1}\mathfrak{c}(u-\tilde{u}, u-\tilde{u}) + \delta_1\|v - \tilde{v}\|_H^2 \\
&\leqslant \Lambda^2\|U - \tilde{U}\|_{\mathcal{H}}^2,
\end{aligned}$$

where we used the hypothesis (**ii**), $\delta_0 = 4(m_b^{-2} + m_c^{-2})M^2$, $\delta_1 = \max\{2m_b^{-2}c_1^2, 2m_c^{-2}c_2^2\}$ and $\Lambda^2 = \max\{\delta_0 d_0^{-1}, \delta_1\}$. Therefore, $\mathcal{F}$ is locally Lipschitz with respect to the second variable.

Now, using the hypothesis (**iii**), we obtain

$$\begin{aligned}
\|\mathcal{F}(t,U)\|_{\mathcal{H}}^2 &\leqslant 2(m_b^{-2} + m_c^{-2})\|F(y-z)\|_{L^2(\Omega)}^2 + 2m_b^{-2}\|f_1(z_t)\|_{L^2(\Omega)}^2 + 2m_c^{-2}\|f_2(y_t)\|_{L^2(\Omega)}^2 \\
&\leqslant 2(m_b^{-2} + m_c^{-2})(|\Omega| + N\|y-z\|_{L^2(\Omega)}^2) + 2m_b^{-2}(|\Omega| + c_3\|z_t\|_{L^2(\Omega)}^2) \\
&\quad + 2m_c^{-2}(|\Omega| + c_4\|y_t\|_{L^2(\Omega)}^2) \\
&\leqslant \delta_2 + \delta_3\|u\|_H^2 + \delta_4\|v\|_H^2 \leqslant \delta_2 + \delta_3 d_0^{-1}\mathfrak{c}(u,u) + \delta_4\|v\|_H^2 \\
&\leqslant \tilde{\Lambda}^2(1 + \|U\|_{\mathcal{H}})^2,
\end{aligned}$$

where $\delta_2 = 6|\Omega|(m_b^{-2} + m_c^{-2})$, $\delta_3 = 4(m_b^{-2} + m_c^{-2})N$, $\delta_4 = \max\{2m_b^{-2}c_3, 2m_c^{-2}c_4\}$ and $\tilde{\Lambda}^2 = \max\{\delta_2, \delta_3 d_0^{-1}, \delta_4\}$. Thus, $\mathcal{F}$ satisfies the sublinear growth.

Finally, as the problem (1.1) is equivalent to (3.9), by Proposition 3.4, Theorem 1.4 [13, p. 185] and by Theorem 11.3.5 [14, p. 261], we conclude that, for all $U_0 \in \mathcal{H}$ there exists a unique global solution $U \in C([0,\infty), \mathcal{H})$. Thus,

$$\begin{aligned}
U \in C([0,\infty), V \times H) &\Rightarrow u \in C([0,\infty), V),\ u_t \in C([0,\infty), H) \\
&\Rightarrow (z,y) \in C([0,\infty), V) \cap C^1([0,\infty), H).
\end{aligned}$$

□

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